An Extension of the Goldbach Property to Semidomains

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General Notation

- $\mathbb{N} := \{1, 2, 3, \ldots\}$.
- $\mathbb{N}_0 := \{0, 1, 2, \ldots\} = \{0\} \cup \mathbb{N}$.
- $\mathbb{P}$ denotes the set of prime numbers.

Origin

Historical Remarks

- The original Goldbach conjecture first appeared in a letter Goldbach sent to Euler in 1742.
- According to the original conjecture, we can write every $n \in \mathbb{N}$ with $n \geq 2$ as the sum of three primes (1 was considered a prime back then).
- The strong version of the Goldbach's conjecture (proposed by Euler) refers to the potential inclusion $2\mathbb{N}_{\geq 2} \subseteq \mathbb{P} + \mathbb{P}$, that is, every even number greater than 2 can be expressed as the sum of two primes).

Significant Progress

Partial but Valuable Progress

- In 1937 Vinogradov proved a result that implies the following: For all $N \in \mathbb{N}$, there exists a positive integer $M \in \mathbb{N}$ such that every positive odd integer $n > M$ can be written as a sum of exactly three primes $p_1, p_2, p_3 > N$.
- The last significant progress towards solving Goldbach conjecture was given by Helfgott, who provided a solution to the weak version of the Goldbach's conjecture, which states that $1 + 2\mathbb{N}_{\geq 3} \subseteq \mathbb{P} + \mathbb{P} + \mathbb{P}$.

Remark. The weak Goldbach conjecture readily follows from the Goldbach conjecture.

Monoids, Units, and Groups

A monoid $M = (M, *)$ is defined as a nonempty set with a binary operation $* : M \times M \to M$ satisfying the following conditions.

- **Associativity**: $(b * c) * d = b * (c * d)$ for all $b, c, d \in M$.
- **Identity Element**: $\exists\ e \in M$ such that $e * b = b * e = b$ for all $b \in M$.

Remark. If the element e in the above definition exists then it is unique. Also, it is often denoted by 0 or 1 if the binary operation of M is written either additively or multiplicative, respectively.

Let M be a monoid.

- $u \in M$ is called a unit or an invertible element if there exists $u^{-1} \in M$ such that $u * u^{-1} = u^{-1} * u = e$
- A group $G = (G, *)$ is defined as a monoid with the additional property that every element in G is a unit.

Commutativity and Cancellativity

Let $(M, *)$ be a monoid.

- **Commutativity:** $(M, *)$ is said to be commutative if $b * c = c * b$ for all $b, c \in M$.
  - **Remark.** We tacitly assume that all monoids are commutative.
- **Cancellativity:** $(M, *)$ is said to be cancellative if $a * c = b * c$ implies $a = b$ for all $a, b, c \in M$.
  - **Remark.** From now on, all monoids referred to here are cancellative.

Examples of Monoids

1. $(\mathbb{N}_0, +)$ is a monoid with identity element 0, but it is not a group because 1 has no inverse.
   - **Notation:** We use $\mathbb{N}_0$ to denote this monoid when it is clear from context.
   - $\mathbb{N}_0$ is referred to as the "prototypical" commutative monoid.
2. $(\mathbb{N}, \cdot)$ is a monoid with identity element 1, but it is not a group because 2 has no inverse.
   - We will use $\mathbb{N}$ to denote this monoid.

   Define, for primes p,

$$\mathbb{N}_0[1/p] := \{\frac{m}{p^n} : m, n \in \mathbb{N}_0\}$$

3. $(\mathbb{N}_0[1/p], +)$ is a monoid with identity element 0.
4. $(\mathbb{N}_0[1/p] \setminus \{0\}, \cdot)$ is a monoid with identity element 1.

Examples of Groups

1. 0 is the only unit of $\mathbb{N}_0$.
    - In general, the identity element is a unit.
2. ± 1 are the units of $(\mathbb{Z} \setminus \{0\}, \cdot)$.
3. $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$, and $(\mathbb{C}, +)$ are groups.
4. $(\mathbb{Q} \setminus \{0\}, \cdot)$, $(\mathbb{R} \setminus \{0\}, \cdot)$, and $(\mathbb{C} \setminus \{0\}, \cdot)$ are groups.

Remark: The set of units of a monoid is actually an abelian group.

A More Interesting Example

Let p be a prime. We will show the units of $M := (\mathbb{N}_0[1/p] \setminus \{0\}, \cdot)$ are exactly p^k, where $k \in \mathbb{Z}$.

1. Notice that p^k is a unit of M as $p^{-k} \in M$ is also an element
2. Other direction
    1. Let $x = \frac{r}{s}$ for $\gcd(r, s) = 1$.
    2. x is an element so s is a power of p
    3. x is a unit so $\frac{1}{x} = \frac{s}{r}$ is also an element, so r is also a power of p
    4. Thus $x = p^k$ for some $k \in \mathbb{Z}$.

Integral Domains

An integral domain is a triple $(R, +, \cdot)$ consisting of a nonempty set and two binary operations satisfying the following:

- $(R, +)$ forms an abelian group with identity 0_R.
- $(R \setminus \{0\}, \cdot)$ is a monoid.
- The identity $a \cdot (b + c) = a \cdot b + a \cdot c$ holds for all $a, b, c \in R$ (the distributive law holds!).
- For all $r, s \in R$, the equality $rs = 0$ implies that $0 \in \{r, s\}$.
- We let $R^* = R \setminus \{0\}$.

Remark. We assume that $0 \neq 1$ in any integral domain R as otherwise R is trivial (i.e., R contains exactly one element).

Examples of Integral Domains

Examples

1. $(\mathbb{Z}, +, \cdot)$ is an integral domain, often called the "prototypical" integral domain.
2. $(\mathbb{Z}/p\mathbb{Z}, +, \cdot)$ for $p \in \mathbb{P}$ is an integral domain.
    - For any $a, b \in \mathbb{Z}/p\mathbb{Z}$, we have that $ab \equiv 0 \pmod{p}$ implies $a \equiv 0 \pmod{p}$ or $b \equiv 0 \pmod{p}$.
3. $(\mathbb{Z}[x], +, \cdot)$ is an integral domain.
4. Similarly, $(\mathbb{Q}[x], +, \cdot)$, $(\mathbb{R}[x], +, \cdot)$, and $(\mathbb{C}[x], +, \cdot)$ are integral domains.

Remark: In a finite integral domain, every nonzero element is a unit.

Fields

A field is a triple $F := (F, +, \cdot)$ that satisfies the following two conditions:

1. F is an integral domain.
2. Every nonzero element of F has a multiplicative inverse.

Remark: This is equivalent to $(F^*, \cdot)$ being a group.

Examples

1. $(\mathbb{Q}, +, \cdot)$ is a field because every nonzero element $r \in \mathbb{Q}$ has multiplicative inverse $\frac{1}{r}$.
2. Similarly, $(\mathbb{R}, +, \cdot)$ and $(\mathbb{C}, +, \cdot)$ are fields.
3. $(\mathbb{Z}/p\mathbb{Z}, +, \cdot)$ is a field for $p \in \mathbb{P}$ since every nonzero residue has a multiplicative inverse modulo p.

Semidomains

A semidomain is a triple $(S, +, \cdot)$ consisting of a nonempty set and two binary operations satisfying the following:

- $(S, +)$ is a monoid.
- $(S^*, \cdot)$ is a monoid.
- S is a subset of an integral domain.

Examples

1. $(\mathbb{N}_0, +, \cdot)$ is the prototypical semidomain.
    - It is a subset of the integral domain $\mathbb{Z}$.
2. $(\mathbb{N}_0[x], +, \cdot)$ is a semidomain
    - It is a subset of the integral domain $\mathbb{Z}[x]$.
3. $(\mathbb{N}_0[1/p], +, \cdot)$ is also a semidomain, where $p \in \mathbb{P}$.

Group of Units

Let S be a semidomain.

- The units of S are the units of the multiplicative monoid $(S^*, \cdot)$.
  - **Remark**: This forms a group known as the group of units

Divisibility and Associates

Let S be a semidomain. We work in S^*.

- An element a is said to divide an element b if there exists an element $c \in S^*$ such that $a \cdot c = b$.
  - This is denoted by $a \mid_S b$.
- Two elements $b, c \in M$ are associates if and only if there exists a unit u such that $b = c \cdot u$. This is equivalent to $b \mid_S c$ and $c \mid_S b$.

Examples

1. $3 \mid_{\mathbb{N}_0} 6$ since $6 = 2 \cdot 3$.
2. In $\mathbb{Z}$, the numbers $-n$ and n are associate for any $n \in \mathbb{N}$ since -1 is a unit.
3. In $\mathbb{N}_0[1/p]$, let $a = m_1 \cdot p^{k_1}$ and $b = m_2 \cdot p^{k_2}$, where $m_i \in \mathbb{N}$, $k_i \in \mathbb{Z}$, and $p \nmid m_i$. Then $a \mid_{\mathbb{N}_0[1/p]} b$ if and only if $m_1 \mid_{\mathbb{N}_0} m_2$.

Irreducible Elements

Let S be a semidomain.

- A nonunit element b of $(S^*, \cdot)$ is called an irreducible element if there do not exist nonunits $c, d \in S^*$ such that $b = c * d$.
- We use $\mathcal{A}(S)$ to denote the set of irreducibles.

Example. $\mathcal{A}(\mathbb{N}_0) = \mathbb{P}$.

An Example

Consider the semidomain $\mathbb{N}_0[1/p]$.

The units are p^k for $k \in \mathbb{Z}$

$\mathcal{A}(\mathbb{N}_0[1/p]) = \{q \cdot p^k : k \in \mathbb{Z}, q \in \mathbb{P}, q \neq p\}$

- We can write every number as $n \cdot p^k$, where $n \in \mathbb{N}$, $p \nmid n$, $k \in \mathbb{Z}$

Example. When $p = 2$, the units are 2^k, and $\frac{5}{4}$ is an irreducible.

Monogenic Semidomains

For each $\alpha \in \mathbb{C}$, we set

$$\mathbb{N}_0[\alpha] := \{f(\alpha) : f(x) \in \mathbb{N}_0[x]\}.$$

Remark: This is the smallest semidomain in $\mathbb{C}$ containing α.

For every element $b \in \mathbb{N}_0[\alpha]$ it has a polynomial representation $f(x) \in \mathbb{N}_0[x]$, where $f(\alpha) = b$.

Examples

- $\alpha \in \mathbb{N}_0$ gives $\mathbb{N}_0$
- $\alpha \in -\mathbb{N}$ gives $\mathbb{Z}$
  - For any $n \in \mathbb{Z}$, take m large enough so that $n - m\alpha \in \mathbb{N}$, and write $(n - m\alpha) \cdot 1 + m \cdot \alpha$.
- $\alpha = \frac{1}{n}$ gives $\{\frac{m}{t} : m \in \mathbb{N}_0, t \in \mathbb{N}, \text{there exists } k \text{ such that } t \mid n^k\}$
- $\alpha = \sqrt{d}$ where $d \in \mathbb{N}$ gives $\mathbb{N}_0[\alpha] = \{b + c\sqrt{d} : b, c \in \mathbb{N}_0\}$

Goldbach Analogues in Commutative Rings

- 1965 D. R. Hayes, "A Goldbach theorem for polynomials with integral coefficients," *Amer. Math. Monthly* **72** (1965) pp. 45–46.Establishes a foundational result: every $f \in \mathbb{Z}[x]$ with $\deg f > 1$ is a sum of two irreducibles.
- 2006 F. Saidak, "On Goldbach's conjecture for integer polynomials," *Amer. Math. Monthly* **113** (2006) pp. 541–545.Extends the Hayes line for integer polynomials and irreducible-summand decompositions.
- 2010 M. Kozek, "An asymptotic formula for Goldbach's conjecture with monic polynomials in $\mathbb{Z}[x]$," *Amer. Math. Monthly* **117** (2010) pp. 365–369.Asymptotically counts representations by monic irreducible summands.
- 2011 P. Pollack, "On polynomial rings with a Goldbach property," *Amer. Math. Monthly* **118** (2011) pp. 71–77.Studies which polynomial rings possess a Hayes-type Goldbach property.

Goldbach Analogues in Semidomains

- 2024 S. Liao and H. Polo, "A Goldbach theorem for Laurent polynomials with positive integer coefficients," *Amer. Math. Monthly* 131(8), 704–711.Initiates the polynomial-semiring program: every non-monomial $f \in \mathbb{N}[x^{\pm 1}]$ with $f(1) > 3$ is a sum of two multiplicative irreducibles.

- 2025 N. Kaplan and H. Polo, "A Goldbach theorem for Laurent series semidomains," *New York J. Math.* 31, 1409–1426.Characterizes semidomains supporting two-summand Laurent-polynomial and three-summand Laurent-series theorems; generalizes Liao–Polo.

The Work of CrowdMath and PRIMES

- PRIMES 2024 E. Li, A. Mopuri, and C. Zhang, "Goldbach theorems for group semidomains." Submitted. Preprint on arXiv: https://arxiv.org/abs/2412.00590Establishes fundamental results in connection to Goldbach analogues in group semidomains
- CrowdMath 2024 F. Gotti, L. Hong, E. Li, H. Liang, H. Polo, and N. Zhang, "Determining the monogenic semidomains having the Goldbach property." Soon to be on arXiv.Determine the monogenic semidomains that satisfy the Goldbach property
- CrowdMath 2025 F. Gotti, M. Gotti, A. Kulkarni, H. Polo, A. Mopuri, R. Wang, and L. Zhou, "On the ascent of the Goldbach properties to semidomains of Laurent series." Soon to be on arXiv.Studies the ascent of Goldbach properties to semidomains and answers an open question in Kaplan-Polo

The Goldbach Properties

Let S be a semidomain. For every $\ell \in \mathbb{N}_0$, define the following:

- Let $G_{\leq \ell}$ denote the set of elements of S that can be expressed as a sum of at most ℓ multiplicative irreducibles of S.

Properties inspired by Goldbach:

1. The weak finitary Goldbach property states that there exists an ℓ such that $S \setminus G_{\leq \ell}$ is finite up to associates.
2. The weak Goldbach property states that there exists an ℓ such that $S \setminus G_{\leq \ell}$ contains only multiplicative units of S.
3. The weak full Goldbach property states that there exists an ℓ such that $S = G_{\leq \ell}$.

Results for Rational Semidomains

We studied when the semidomain $S_q := \mathbb{N}_0[q]$ is Goldbach for $q \in \mathbb{Q}_{>0} \setminus \mathbb{N}$.

Theorem

The following statements are equivalent:

1. S_q *has the weak finitary Goldbach property*
2. S_q *has the weak Goldbach property*
3. S_q *has the weak full Goldbach property*
4. $q \in \mathbb{N}_{\geq 2}^{-1}$

Sketch of Proof

Proof Sketch: Let $q = \frac{1}{n}$.

1. The elements of S_q are of the form $\frac{m}{t}$, where $\gcd(m, t) = 1$ and $t \mid n^k$ for some positive integer k, and the units are of the form $\frac{1}{t}$.
2. Notice that p is an irreducible whenever $p \nmid n$.
3. After multiplying the numerator by sufficiently large powers of n, use Vinogradov's theorem (large odd is $p + q + r$) by choosing primes $p > n$ and adjust for parity. This gives a decomposition into at most 4 irreducible elements.

Algebraic Numbers

Let $\alpha \in \mathbb{C}$.

- α is an algebraic number if it is the root of a nonconstant integer polynomial $f(x) \in \mathbb{Z}[x]$.
  - Otherwise, it is transcendental (e.g. π, e)
- If α is algebraic, then there is a unique monic irreducible polynomial $m_\alpha(x) \in \mathbb{Q}[x]$, called the minimal polynomial of α, with the following properties:
  - $m_\alpha(\alpha) = 0$, and
  - For any polynomial $f(x) \in \mathbb{Q}[x]$ such that $f(\alpha) = 0$, then $m_\alpha(x) \mid_{\mathbb{Q}[x]} f(x)$.
- If $m_\alpha(x) \in \mathbb{Z}[x]$ then α is called an algebraic integer.

Remark. The set of algebraic integers is an integral domain.

Remark. The set of algebraic numbers is a field and is denoted $\mathbb{A}$.

Algebraic Numbers (Examples)

Examples

- $\mathbb{Q}$
- $\frac{\sqrt{3}}{3}$, with minimal polynomial $x^2 - \frac{1}{3}$
- $\sqrt{2}$ is an algebraic integer, with minimal polynomial $x^2 - 2$
- $\sqrt{2} + \sqrt{3}$ is also an algebraic integer, with minimal polynomial $x^4 - 10x^2 + 1$

Goldbach Properties for Algebraic Semidomains

Question. Let $\alpha > 0$ be an irrational algebraic number. When does $S_\alpha := \mathbb{N}_0[\alpha]$ have the Goldbach property?

Theorem

If $\mathbb{N}_0[\alpha]$ has the weak finitary Goldbach property then α is a multiplicative unit.

Here are some non-examples:

- $\mathbb{N}_0[\sqrt{2}]$
- $\mathbb{N}_0[\phi]$

Goldbach Properties for Algebraic Semidomains (continued)

Proof Sketch. We show the contrapositive by proving that infinitely many non-expressible elements exist

1. As α is not a multiplicative unit, there are no multiplicative units.
2. For each $n \in \mathbb{N}$, let $f(n)$ denote the largest positive integer such that $n\alpha - f(n) \in S_\alpha$.
3. The additive divisors of $n\alpha - f(n)$ are all divisible by α
    - If polynomial representation has a nonzero constant term then would use $f(n) + 1$
4. Thus every irreducible additive divisor of $n\alpha - f(n)$ is α.
5. Taking all $n \in \mathbb{N}$, since α is irrational then $n\alpha - f(n)$ is a set of infinitely many elements in S_α that cannot be written as a sum of at most ℓ irreducibles
    - Remove $k\alpha$ for $0 \leq k \leq \ell$

Goldbach Properties for Algebraic Semidomains (continued)

Question. What about the other direction?

Theorem

Let α be an algebraic number such that α is a multiplicative unit of $\mathbb{N}_0[\alpha]$. Then for almost all α, S_α has weak full Goldbach property.

Examples

- $\mathbb{N}_0[\phi - 1]$ is weak full Goldbach
- $\mathbb{N}_0[\alpha]$ is weak full Goldbach, where α is the positive root of $x^3 + x = 1$

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End of Presentation

Thank you for your time!

Goldbach Properties for Algebraic Semidomains (continued)

Question. What about the other direction?

Theorem

Let α be an algebraic number such that α is a multiplicative unit of $\mathbb{N}_0[\alpha]$. Let d denote the GCD of the exponents of $m_\alpha(x)$ with nonzero coefficients. Let n denote the degree of $\beta = \alpha^d$. If the Galois group of β contains an n-cycle, then S_α has weak full Goldbach property.

Examples

- $\mathbb{N}_0[\phi - 1]$ is weak full Goldbach
- $\mathbb{N}_0[\alpha]$ is weak full Goldbach, where α is the positive root of $x^3 + x = 1$

On the Ascent of the Goldbach Property to Semidomains of Laurent Series

(joint work with Felix Gotti, Marly Gotti, Leo Hong, Advaith Mopuri, Harold Polo, Richard Wang, and Lawrence Zhou)

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Outline

Preliminary Definitions

Throughout, $S^\times$ denotes the group of multiplicative units of S^*.

Definition. $\mathscr{A}_+(S)$ denotes the additive atoms of S, that is, the atoms of the monoid $(S, +)$. These are not the multiplicative irreducibles $\mathscr{A}(S)$ from the previous talk.

Definition.

- A monoid is atomic if every nonunit is a finite product of atoms, and Furstenberg if every nonunit is divisible by at least one atom. Every atomic monoid is Furstenberg.
- We call S additively atomic (resp., additively Furstenberg) when $(S, +)$ is.

Definition.

- A submonoid N of M is divisor-closed if $d \mid_M m$ and $m \in N$ imply $d \in N$; two elements are associates if each divides the other.
- A semifield is a semidomain in which every nonzero element is a unit, such as $\mathbb{Q}_{\geq 0}$.

Laurent Polynomials and Laurent Series

Definition. Let S be a semidomain, and let R be an integral domain containing S as a subsemiring.

- The polynomial semiring $S[x]$ is the subsemiring of $R[x]$ consisting of the polynomials with coefficients in S.
- The Laurent polynomial semiring $S[x^{\pm 1}]$ is the subsemiring of $R[x^{\pm 1}]$ consisting of the Laurent polynomials with coefficients in S.
- The Laurent series semiring $S[\![x^{\pm 1}]\!]$ is the subsemiring of $R[\![x^{\pm 1}]\!]$ consisting of the Laurent series with coefficients in S.

Every Laurent series considered here has only finitely many negative terms.

Supports

Definition. For $f = \sum_{i \geq 0} c_i x^{k_i} \in S[\![x^{\pm 1}]\!]$, the support of f is

$$\operatorname{supp}(f) := \{k_i : c_i \neq 0,\ i \in \mathbb{N}_0\},$$

and f is a monomial, binomial, or trinomial when $|\operatorname{supp}(f)|$ is 1, 2, or 3.

Definition. A semidomain S is additively reduced if 0 is the only invertible element of $(S, +)$; equivalently, $a + b = 0$ forces $a = b = 0$.

Hence no additive cancellation occurs, and so

$$\operatorname{supp}(g) + \operatorname{supp}(h) \subseteq \operatorname{supp}(gh).$$

Divisor-Closed Submonoids

Fact. Suppose that S is additively reduced. Then S^* and $S[x^{\pm 1}]^*$ are divisor-closed submonoids of $S[\![x^{\pm 1}]\!]^*$.

Consequently,

$$\mathscr{A}(S) \subseteq \mathscr{A}(S[x^{\pm 1}]) \subseteq \mathscr{A}(S[\![x^{\pm 1}]\!]),$$

$$S[\![x^{\pm 1}]\!]^\times = S[x^{\pm 1}]^\times = \{ux^k : u \in S^\times \text{ and } k \in \mathbb{Z}\}.$$

Irreducibility therefore extends across S, $S[x^{\pm 1}]$, and $S[\![x^{\pm 1}]\!]$, so we simply say irreducible.

Goldbach Theorems for Laurent Polynomials and Series

Theorem (Liao–Polo, 2024)

Every $f \in \mathbb{N}_0[x^{\pm 1}]$ with $f(1) > 3$ and $|\text{supp}(f)| > 1$ can be written as the sum of two irreducibles.

Theorem (Kaplan–Polo, 2025)

Let S be an additively reduced and additively atomic semidomain with $\mathscr{A}_+(S) = S^\times$. Then every nonzero non-monomial Laurent series in $S[\![x^{\pm 1}]\!]$ can be written as the sum of at most three multiplicative irreducibles.

Examples.

- Note that $S = \mathbb{N}_0$ works, as $\mathscr{A}_+(\mathbb{N}_0) = \{1\} = \mathbb{N}_0^\times$; thus, every non-monomial series in $\mathbb{N}_0[\![x^{\pm 1}]\!]$ is a sum of at most three irreducibles.
- But, $S = \mathbb{N}_0[\sqrt{2}]$ fails by virtue of the fact that $\sqrt{2} \in \mathscr{A}_+(S) \setminus S^\times$.

Our Goal

Improve the bound from three to two, thereby settling a conjecture of Kaplan and Polo (first proposed three years ago, in the original version of their paper):

Conjecture (Kaplan–Polo, 2025)

Let S be an additively reduced and additively Furstenberg semidomain. The following are equivalent.

- $\mathscr{A}_+(S) \subseteq S^\times$.
- Every nonzero non-monomial $f \in S[\![x^{\pm 1}]\!]$ is the sum of at most two multiplicative irreducibles.
- There is $m \in \mathbb{N}$ with $m \geq 2$ such that every nonzero non-monomial $f \in S[\![x^{\pm 1}]\!]$ is the sum of at most m multiplicative irreducibles.

The Goldbach Property

Definition. Let S be an additively reduced semidomain. We say that S is a Goldbach semidomain if, for every nonzero nonunit $s \in S$, the element $2s$ can be written as the sum of two multiplicative irreducibles.

- We call a nonzero nonunit of the form $2s$ even, and a nonzero nonunit that is not even is odd.
- The classical Goldbach conjecture says exactly that $\mathbb{N}_0$ is a Goldbach semidomain.

Definition. S is a strong Goldbach semidomain if every nonzero nonunit of S can be written as the sum of two multiplicative irreducibles; equivalently, if every nonzero nonunit lies in $G_{\leq 2}$.

So Goldbach asks of the even elements exactly what strong Goldbach asks of all of them.

Fact. Every strong Goldbach semidomain is Goldbach.

The Goldbach Property Does Not Always Hold

Example. Let k and n be relatively prime positive integers with $1 < k < n$, and set $S := \mathbb{N}_0 \left[\frac{k}{n}\right]$.

- S is a subsemiring of $\mathbb{R}_{\geq 0}$, so it is additively reduced, and $S^\times = \{1\}$.
- Proved in CrowdMath 2024, its multiplicative irreducibles are strictly bounded above by

$$c = n^2 \frac{k-1}{n-k},$$

  and since $k \geq 2$ and $n > k$ we have $c \geq \frac{n^2}{n-k} > 1$.
- Because $\mathbb{N}_0 \subset S$, the semidomain is unbounded, so we may pick $s \in S$ with $s \geq c$. Such an s satisfies $s > 1$ and is therefore a nonzero nonunit, and $2s \geq 2c$.
- The sum of any two irreducibles of S is strictly less than $2c$, so $2s$ is not the sum of two irreducibles.

Fact. Irreducibles being bounded implies the Goldbach property cannot occur.

Goldbach But Not Strong Goldbach

Example. Let S consist of all nonnegative rationals whose numerator is a nonnegative integer and whose denominator is a positive odd integer. Then

$$S^\times = \left\{ \frac{2n+1}{2m+1} : n, m \in \mathbb{N}_0 \right\},$$

and 2 together with its associates are the only multiplicative irreducibles of S.

- S is not strong Goldbach: the nonunit 2 is not the sum of two irreducibles.
- S is Goldbach. For a nonzero nonunit $n/d \in S$ we have $2 \mid n$. Take $t \in \mathbb{N}$ with $2^t \mid n$ and $2^{t+1} \nmid n$, and set $u := n/2^t$, so that $u/d \in S^\times$. Then

$$2\left(\frac{n}{d}\right) = 2^{t+1}\left(\frac{u}{d}\right) = 2\left(\frac{u(2^t-1)}{d}\right) + 2\left(\frac{u}{d}\right),$$

  where $u(2^t-1)/d$ and u/d are units, so both summands are irreducible.

Tools for Monolithic Series and Rescaling Exponents

As stated earlier, we aim for the bound of two for Laurent series over additively Furstenberg semidomains.

Definition. A nonzero Laurent series $f \in S[\![x^{\pm 1}]\!]$ is monolithic if every factorization $f = g \cdot h$ forces g or h to be a monomial in $S[\![x^{\pm 1}]\!]$.

Lemma (GGHKMPWZ, 202?)

Let S be an additively reduced semidomain, and let $f \in S[\![x^{\pm 1}]\!]$ with $|\operatorname{supp}(f)| \geq 2$ and $\min \operatorname{supp}(f) = 0$.

- If $\operatorname{supp}(f)$ contains exactly one odd integer and $|\operatorname{supp}(f)| > 3$, then f is monolithic.
- If $\operatorname{supp}(f)$ does not contain positive even integers, then f is monolithic.

Corollary (GGHKMPWZ, 202?)

Write $f(x^{1/2})$ for the series obtained by the substitution $x^k \mapsto x^{k/2}$. If $f(x^{1/2}) \in S[\![x^{\pm 1}]\!]$ is the sum of two irreducibles, then so is f. We may therefore scale exponents freely.

The Semifield Case

Proposition (GGHKMPWZ, 202?)

For an additively reduced semifield F, the semidomain $F[\![x^{\pm 1}]\!]$ is strong Goldbach.

Proof sketch. Small supports split into binomials.

- Thus, we assume $|\mathrm{supp}(f)| \geq 5$, and write $f = \sum_{i \geq 0} c_i x^i$ with $c_0 \in F^*$, and scale (by an appropriate power) so that $\mathrm{supp}(f)$ contains at least two odd integers and at least one positive even integer. Let $2k+1$ be the smallest odd integer in $\mathrm{supp}(f)$.
- We then split f by parity, halving c_0 between the two pieces:

$$g := \frac{c_0}{2} + \frac{c_{2k+1}}{2} x^{2k+1} + \sum_{i \geq 1} c_{2i} x^{2i}$$

$$h := \frac{c_0}{2} + \frac{c_{2k+1}}{2} x^{2k+1} + \sum_{i \geq k+1} c_{2i+1} x^{2i+1}.$$

- Then $f = g + h$, whence both summands are irreducible by the Lemma.

Main Theorem

Theorem (GGHKMPWZ, 202?)

Let S be an additively reduced and additively Furstenberg semidomain. The following are equivalent.

- $\mathscr{A}_+(S) \subseteq S^\times$.
- Every nonzero non-monomial $f \in S[\![x^{\pm 1}]\!]$ is the sum of at most two multiplicative irreducibles.
- There is $m \in \mathbb{N}$ with $m \geq 2$ such that every nonzero non-monomial $f \in S[\![x^{\pm 1}]\!]$ is the sum of at most m multiplicative irreducibles.

If these hold, then every nonzero non-monomial $f \in S[\![x^{\pm 1}]\!]$ is the sum of exactly two irreducibles unless

- $f = c_0 x^{k_0} + c_1 x^{k_1}$ with $c_0, c_1 \in S^*$ and $k_0 > k_1$, where $c_0 \in S^\times$ or $c_1 \in S^\times$, or
- $f = c_0 x^{k_0} + c_1 x^{k_1} + c_2 x^{k_2}$ with $c_0, c_1, c_2 \in S^\times$ and $k_0 > k_1 > k_2$.

This settles the conjecture of Kaplan and Polo and improves their bound from three to two!

What Is Still Open

Establishing that the Goldbach property descends from $S[\![x^{\pm 1}]\!]$ to S is straightforward, but the ascent is not.

We have established some sufficient conditions for the ascent (read our paper to learn about them).

But, this is not enough to solve the following open question:

Question

Let S be a Goldbach (resp., strong Goldbach) semidomain. Find natural and verifiable conditions on S ensuring that $S[\![x^{\pm 1}]\!]$ is also Goldbach (resp., strong Goldbach).

References


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End of Presentation

Thank you for your time!